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AN ANALYSIS OF THREE-MOVE FINITE GAMES

M. Dresher, O. Helmer and R. Wagner

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## AN ANALYSIS OF THREE-MOVE FINITE GAMES

M. Dresner, O. Helmer and R. Wagner

1. Introduction.

The complexity of possible information patterns increases with the number of moves available to the players. If we assume that a move is made with either complete or no knowledge about the other move, a 2-move game has only two possible patterns of information. Either there is perfect information, and thus one move precedes the other, or there is no information, in which case the moves are in effect made simultaneously. In each case the game has a value and optimal strategies, -- a pure strategy in the case of perfect information, and generally mixed strategies if the moves are made simultaneously.

A 3-move game, i.e., one in which two moves are made by one player and one by the other, introduces many additional types of information patterns. Again, we assume that information, if any, about a preceding move is complete. At the time of his second move, the player with two moves may have forgotten his first move or the information known at this first move. The effect of this introduction of imperfect recall is to yield a game without a value. Further, in these cases it is generally impossible to express the 3-move game in normal form.

2. Three-Move Games.

We consider the zero-sum two-person game in which a total of three moves are made by the players. Each move is a choice from a finite set. There is no loss of generality if we assume that Player I has two moves, the choices of  $i$  and  $\bar{i}$ , and that Player II has one move, the choice of  $j$ . It is also assumed that the choice  $i$  is made not later than the choice  $\bar{i}$ .

There are fourteen essentially different patterns of information for any 3-move game. Of these, seven can be classified as perfect recall -- i.e. Player I in choosing  $\bar{I}$  always knows both his previous choice  $i$  and the information known at the time  $i$  was selected. Using the functional notation  $j(i)$  to indicate that Player II makes his choice  $j$  knowing Player I's choice  $i$ , we obtain the following seven cases of perfect recall:

1.  $i \quad \bar{I}(i) \quad j(i, \bar{I})$
2.  $i \quad \bar{I}(i) \quad j(i)$
3.  $i \quad \bar{I}(i) \quad j(\bar{I})$
4.  $i \quad \bar{I}(i) \quad j$
5.  $i \quad j(i) \quad \bar{I}(i, j)$
6.  $i \quad j \quad \bar{I}(i, j)$
7.  $j \quad i(j) \quad \bar{I}(i, j).$

If we assume that Player I's second move is made with imperfect recall of his first move, we obtain the following additional seven patterns of information:

8.  $i \quad \bar{I} \quad j(i, \bar{I})$
9.  $i \quad \bar{I} \quad j(i)$
10.  $i \quad \bar{I} \quad j$
11.  $i \quad j(i) \quad \bar{I}(j)$
12.  $i \quad j \quad \bar{I}(j)$
13.  $j \quad i(j) \quad \bar{I}(j)$
14.  $j \quad i(j) \quad \bar{I}(i).$

In each of the cases 8 through 13, Player I does not know his first move  $i$  at the time he makes his second move  $\bar{i}$ . The final case, 14, may require "signalling", since Player I has forgotten his opponent's move but remembers his own first move.

The following table presents the solution of each of the fourteen types of three-move games. If a game does not have a value, then the solutions of the minorant and majorant games (in terms of so-called behavior strategies) are given, as well as their respective values  $\underline{v}$  and  $\bar{v}$ . The strategy at move  $i$  is represented by a vector  $X$  such that each  $x_i \geq 0$ ,  $\sum_1 x_i = 1$ ; and similarly the strategies at moves  $\bar{i}$  and  $j$  are denoted respectively by vectors  $\bar{X}$  and  $Y$ .

Three-Move Games

<u>Case</u>	<u>Move and information</u>		<u>Value</u>
1.	1	$\bar{1}(1)$ $J(1, \bar{1})$	$\max_i \max_{\bar{1}} \min_j a_{i\bar{1}j}$
2.	1	$\bar{1}(1)$ $J(1)$	$\max_i \max_{\bar{1}} \min_Y \sum_{\bar{1}} \sum_j a_{i\bar{1}j} \bar{x}_1 y_j$
3.	1	$\bar{1}(1)$ $J(\bar{1})$	$\max_i \max_X \min_Y \sum_{\bar{1}} \sum_j a_{i\bar{1}j} x_1 y_j$ [1]
4.	1	$\bar{1}(1)$ $J$	$\max_W \min_Y \sum_k \sum_j a_{kj} v_k y_j$ [2]
5.	1	$J(1)$ $\bar{1}(1, J)$	$\max_i \min_j \max_{\bar{1}} a_{i\bar{1}j}$
6.	1	$J$ $\bar{1}(1, J)$	$\max_X \min_Y \sum_{\bar{1}} \sum_j (\max_{\bar{1}} a_{i\bar{1}j}) x_1 y_j$
7.	$J$	$1(J)$ $\bar{1}(1, J)$	$\min_j \max_i \max_{\bar{1}} a_{i\bar{1}j}$
8.	1	$\bar{1}$ $J(1, \bar{1})$	same as case 1 [3]
9.	1	$\bar{1}$ $J(1)$	same as case 2 [3]
10.	1	$\bar{1}$ $J$	$\begin{cases} \bar{v} = \max_X \max_{\bar{1}} \min_Y \sum_{\bar{1}} \sum_j \sum_j a_{i\bar{1}j} x_1 \bar{x}_1 y_j \\ \bar{v} = \min_Y \max_X \max_{\bar{1}} \sum_{\bar{1}} \sum_j \sum_j a_{i\bar{1}j} x_1 \bar{x}_1 y_j \end{cases}$
11.	1	$J(1)$ $\bar{1}(J)$	same as case 5 [3]
12.	1	$J$ $\bar{1}(J)$	$\begin{cases} \bar{v} = \max_{\bar{1}(J)} \max_X \min_Y \sum_{\bar{1}} \sum_j a_{i\bar{1}j} x_1 y_j \\ \bar{v} = \min_Y \max_{\bar{1}(J)} \max_X \sum_{\bar{1}} \sum_j a_{i\bar{1}j} x_1 y_j \end{cases}$
13.	$J$	$1(J)$ $\bar{1}(J)$	same as case 7 [3]
14.	$J$	$1(J)$ $\bar{1}(1)$	$\begin{cases} \bar{v} = \max_U \min_j \max_i \sum_s u_s a_{i\bar{1}s} \\ \bar{v} = \min_Y \max_{\bar{1}(1)} \sum_j (\max_{\bar{1}} a_{i\bar{1}j}) y_j \end{cases}$ [4]

[1] This case is essentially case 2, with the roles of 1 and  $\bar{1}$  interchanged.

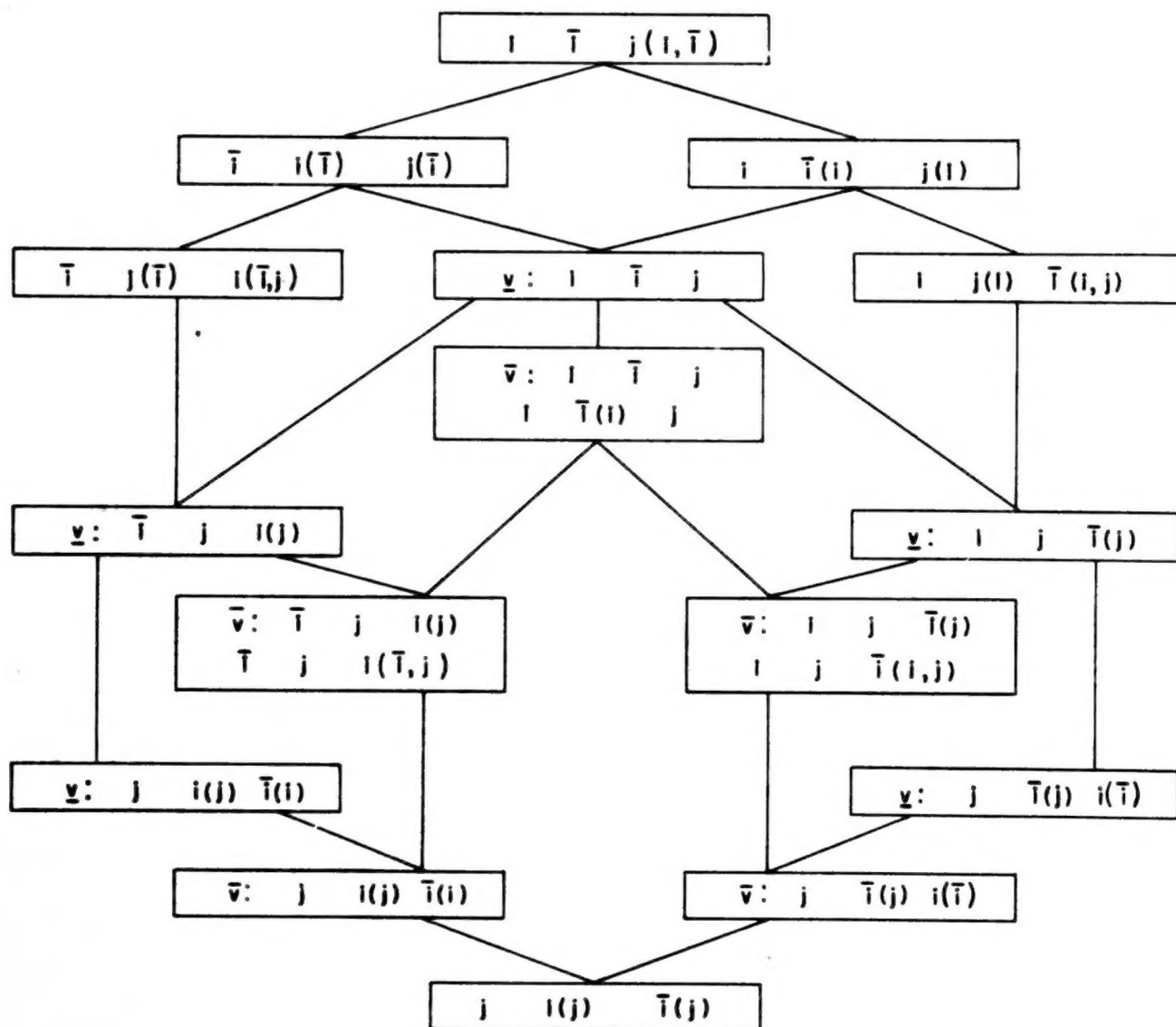
[2] Each  $k$  represents one of the pairs of 1,  $\bar{1}$  (in the later example these are the four pairs: 1, 1; 1, 2; 2, 1; 2, 2).

[3] Since at move 1 there is a pure strategy solution, it may be assumed that Player I knows move 1 at the time of move  $\bar{1}$ .

[4] Each  $\bar{1}_s$  represents one of the possible ways of choosing  $\bar{1}$  as a function of 1 (in the example these are:  $\bar{1} = 1, 2, 1, 3-1$ ).

The cases of imperfect recall were solved under the assumption that Player I's two moves could result from mutual strategies, in the sense that if there were several pairs of optimal strategies for his two moves, one pair of strategies could have been chosen in advance of the play of the game.

The following diagram shows the partial order of the game values of the essentially different cases, in each instance the smaller value being associated with the upper end of the connecting line segment. For each pair of cases in the diagram where the partial order establishes no ordering, examples for both ways of ordering can be found.



3. Example.

As an example, consider the game where each move consists of two choices, with the following payoff matrix:

$i \backslash j$		1	2
1	2		
1	1	2	-3
1	2	-2	1
2	1	-3	1
2	2	-1	2

The following table lists the solutions of this example.



Solution of Example

Case	Move and information	Value	$\bar{x}$	$\bar{y}$	$\bar{z}$
1.	1 $\bar{i}(1)$ $j(1, \bar{i})$	-1	(0,1)	(0,1)	(0,1) if $i = \bar{i} = 1$ (1,0) otherwise
2.	1 $\bar{i}(1)$ $j(1)$	$-\frac{1}{2}$	(1,0)	$(\frac{3}{8}, \frac{5}{8})$ if $i=1$ (0,1) if $i=2$	$(\frac{1}{2}, \frac{1}{2})$ if $i=1$ (1,0) if $i=2$
4.	1 $\bar{i}(1)$ $j$	$\frac{1}{8}$	$w = (\frac{3}{8}, 0, \frac{5}{8}, 0)$		$(\frac{5}{8}, \frac{3}{8})$
5.	1 $j(1)$ $\bar{i}(1, j)$	1	(1,0)	(1,0) if $i=j=1$ (0,1) otherwise	(0,1) if $i=1$ (1,0) if $i=2$
6.	1 $j$ $\bar{i}(1, j)^{-}$	$\frac{5}{4}$	$(\frac{3}{4}, \frac{1}{4})$	(1,0) if $i=j=1$ (0,1) otherwise	$(\frac{1}{4}, \frac{3}{4})$
7.	$j$ $i(j)$ $\bar{i}(1, j)$	2	(0,1) if $j=1$ (1,0) if $j=2$	(0,1) if $j=1$ (1,0) if $j=2$	(y, 1-y)
10.	1 $\bar{i}$ $j$	$\bar{y} = -\frac{4}{9}, \bar{v} = \frac{1}{8}$	$(\frac{7}{9}, \frac{2}{9})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{5}{8}, \frac{3}{8})$
12.	1 $j$ $\bar{i}(j)$	$\bar{y} = \frac{7}{6}, \bar{v} = \frac{5}{4}$	$(\frac{5}{6}, \frac{1}{6})$	(1,0) if $j=1$ (0,1) if $j=2$	$(\frac{1}{4}, \frac{3}{4})$
14.	$j$ $i(j)$ $\bar{i}(1)$	$\bar{y}=2, \bar{v}=2$ [1]	(1,0) if $j=1$ (0,1) if $j=2$	(1,0) if $i=1$ (0,1) if $i=2$	(y, 1-y)

<sup>1</sup> In general,  $\bar{y} \neq \bar{v}$ .

Case 14, the signalling case, is the only one in which joint randomization would lead to something new. Any other case of imperfect recall, under the assumption of joint randomization, could be put into normal form and thereby would become identical with the corresponding case of perfect recall.<sup>1</sup> Case 14 for the following revised payoff matrix

		J	
		1	2
I	1	1 1	2 -3
	2	1 2	-2 2
	1	2 1	-3 1
	2	2 2	-1 0

has the solution

$$\underline{y} = 1, \quad \underline{x} = \begin{cases} (1,0) & \text{if } j = 1 \\ (0,1) & \text{if } j = 2 \end{cases} \quad \text{and } \bar{x} = (1,0),$$

$$\bar{y} = \frac{5}{8}, \quad \bar{y} = (\frac{1}{8}, \frac{7}{8}).$$

However, if joint randomization were allowed, the value  $\frac{3}{4}$  could be obtained by Player I by the mixture  $(\frac{3}{4} A + \frac{1}{4} B)$  of the following pair of strategies:

	I		$\bar{I}$
	1	2	
A	1 if $j=1$	2 if $j=2$	1
	2 if $j=1$	1 if $j=2$	2

<sup>1</sup> This is a consequence of Lemma 4 in a paper by Krentel, McKinsey, and Quine, A Simplification of Games in Extensive Form, to be published in the Duke Mathematical Journal.

#### 4. Chance Moves.

If one of the three moves is a chance move and one move is made by each of the players, then thirteen cases of different information patterns occur, in view of the fact that the probability distribution for the chance moves may be a function of a player's move:

1. c 1 J
2. c 1(c) J
3. c 1(c) J(c)
4. c 1 J(1)
5. c 1(c) J(1)
6. c 1 J(c,1)
7. 1 c(1) J
8. 1 c(1) J(1)
9. 1 c(1) J(c)
10. 1 c(1) J(c,1)
11. 1 J c(1,J)
12. 1 J(1) c(1,J)
13. J 1(J) c(1)

Each such game has a value and optimal strategies, since it can be put into normal form and solved essentially as a 2-move game. The details will not be given here.